

REASONING UNDER UNCERTAINTY:
BAYESIAN APPROACH

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Introduction

- Probability is a mathematical approach for processing uncertain information.
- Probability theory proposes the existence of a number $P(E)$ called the probability, which is the likelihood of some event E occurring from a random experiment
- The set of all possible outcomes of an experiment is called **sample space**, denoted as S
- Each possible outcome of the experiment is an event E and is part of sample space

Introduction

- Discrete Sample Space

Sample space for rolling a die:

$$S = \{1, 2, 3, 4, 5, 6\}$$

One way of determining the probability is to perform experiment by rolling the die N times, record number of wins for particular event $W(E)$.

$$\text{Thus, } P(E) = W(E)/N$$

- Also known as **posteriori probability** meaning “after the event”

Introduction

Compound Probabilities

- Combination of more than 1 events happening

Intersection

$$P(A \cap B) = P(A) * P(B)$$

Example

Rolling a die with odd number and also one divisible by 3.

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{1, 3, 5\}$$

$$B = \{3, 6\}$$

$$A \cap B = \{3\}$$

$$P(A) = 3/6 = 1/2$$

$$P(B) = 2/6 = 1/3$$

$$P(A \cap B) = 1/2 * 1/3 = 1/6$$

Introduction

Union

- Determining either one happening

$$P(A \text{ or } B) = P(A \cup B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Consider for example the probability of rolling an odd-numbered die (event A) or one divisible by 3 (event B)

$$P(A \cup B) = 1/2 + 1/3 - 1/2 * 1/3 = 2/3$$

Introduction

Conditional Probabilities

- Previous example deals with independent events
- Events that are not mutually exclusive will influence one another
- The probability of an event A occurring, given that an event B has already occurred, is called **conditional probability**,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Introduction

Conditional Probabilities (Example)

Consider previous example again:

Probability of rolling the number 3 (event A) given that we know that one divisible by 3 (event B) has occurred i.e.

$$P(A \cap B) = 1/6$$

$$P(B) = 2/6$$

$$P(A|B) = \frac{1/6}{2/6} = 1/2$$

Bayesian Approach

- Used posteriori probability which is the reverse of conditional probability
- Conditional Probability, $P(A|B)$ is concerned with the probability of event A given event B has occurred.
- Whereas, Posteriori Probability concerned with the probability of **earlier event** given that some **later event** has occurred.
- Typical problem in machine diagnostic.
 - **Later Event** --> machine fault **symptoms**
 - **Earlier Event** --> the **cause** of the symptoms
- Conditional Probability **forward** in time
- Posterior Probability **backward** in time.

Bayesian Approach

- Formal definition of **Bayes Theorem** provides the probability of the truth of some hypothesis H, given the evidence E:

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$

$P(H|E)$ = probability that H is true given evidence E

$P(H)$ = probability that H is true

$P(E|H)$ = probability of observing E when H is true

$P(E)$ = probability of E.

Bayesian Approach

- P(E) can also be written as:

$$P(E) = P(E|H)*P(H) + P(E|\sim H)*P(\sim H)$$

P(E|~H) = probability that E is true when H is false

P(~H) = probability that H is false

- Therefore, another form of Bayes Theorem substituting P(E) with the above:

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E|H)*P(H) + P(E|\sim H)*P(\sim H)}$$

Example 1

- Dua orang calon akan bertanding di suatu kawasan. Kebarangkalian calon yang pertama akan dipilih ialah 0.7, kebarangkalian calon yang kedua akan dipilih ialah 0.3. Jika yang pertama dipilih, kebarangkalian sebuah sekolah baru akan dibina ialah 0.60; tetapi jika yang kedua dipilih, kebarangkaliannya ialah 0.4. Apakah kebarangkalian:
 - a. Sebuah sekolah baru akan dibina?
 - b. Calon pertama dipilih jika sekolah baru telah dibina?
 - c. Calon kedua dipilih jika sekolah baru tidak dibina?

Example 2

- 80% daripada pelajar yang memohon kerja melalui Unit Kerjaya adalah pelajar sarjana manakala 20 peratus adalah sarjana muda. Seseorang pelajar sarjana yang memohon kerja melalui unit kerjaya mempunyai 60% peluang untuk mendapat pekerjaan, sedangkan pelajar sarjana muda mempunyai peluang sebanyak 40% sahaja. Cari kebarangkalian:
 - a. seorang pelajar (dipilih secara rawak) yang memohon kerja melalui Unit Kerjaya akan mendapat pekerjaan
 - b. Jika diketahui seseorang pelajar telah mendapat kerja, apakah kebarangkalian dia seorang pelajar ijazah sarjana

Bayesian Approach

- Bayes need Prior Probabilities of an event to interpret present situation.
- Bayes can be used in rule based ES with the typical structure:
IF E THEN H
- or inference chains of the form
IF X THEN E → IF E THEN H
- Bayes theorem can also find probability hypothesis H not true (H complement) given the evidence using the following equation:

$$P(\sim H|E) = \frac{P(\sim H) * P(E|\sim H)}{P(E)}$$

Bayesian Approach

Example 1:

ES express knowledge in IF-THEN format:

IF X is true
THEN Y can be concluded with probability p

If we observed that X is true, then we can conclude that Y exists with the specified probability, for eg.:

IF the patient has a cold (earlier event/hypothesis H)
THEN the patient will sneeze (0.75) (later event/evident E)

If reason abductively, observe Y (patient sneeze) while knowing nothing about X (patient has cold).

What can we conclude??

Bayesian Approach

- Using Bayes Theorem we can find probability of X.
In this case $X = H$ (hypothesis) and $Y = E$ (evidence).
- Consider whether a person has a cold (hypothesis) given he sneezes (evidence).
- Translate this problem into Bayes Theorem

$$P(H|E) = \frac{P(H) * P(E|H)}{P(E)}$$

Bayesian Approach

- Suppose we know, (Prior Probabilities)

$$\begin{aligned} P(H) &= P(\text{has a cold}) \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(E|H) &= P(\text{observed sneezing} \mid \text{has a cold}) \\ &= 0.75 \text{ (from the rule)} \end{aligned}$$

$$\begin{aligned} P(E|\sim H) &= P(\text{observed sneezing} \mid \text{does not have a cold}) \\ &= 0.2 \end{aligned}$$

then,

$$\begin{aligned} P(E) &= P(\text{was observed sneezing}) \\ &= \mathbf{P(E|H)*P(H) + P(E|\sim H)*P(\sim H)} \\ &= P(0.75)(0.2) + (0.2)(0.8) \\ &= 0.31 \end{aligned}$$

Bayesian Approach

- Therefore:

$$\begin{aligned} P(H|E) &= P(\text{has a cold}|\text{was observed sneezing}) \\ &= \frac{(0.75)*(0.2)}{0.31} \\ &= 0.48387 \end{aligned}$$

Could also determine, what is the probability of having a cold given he was not sneezing.

$$\begin{aligned} P(H|\sim E) &= \frac{P(H) * P(\sim E|H)}{P(\sim E)} \\ &= \frac{(0.2) * (1 - 0.75)}{(1 - 0.31)} \\ &= 0.07246 \end{aligned}$$

Bayesian Approach

Example 2:

- Patient with chest pains are often given ECG test. Test result are classified either +ECG suggesting heart disease (+HD) and -ECG \rightarrow -HD.
- Assuming patient has produces a +ECG, what is the probability he has HD i.e. $P(+HD|+ECG)$.

Bayesian Approach

Assume the following priori probability: (from sample of previous patient)

$$P(+HD) = 0.1 \quad P(-HD) = (1 - P(+HD)) = 0.9$$

$$P(+ECG|+HD) = 0.9 \quad P(-ECG|-HD) = 0.95$$

$$P(+ECG|-HD) = 1 - P(-ECG|-HD) = 0.05$$

Therefore, using Bayes's Theorem:

$$\begin{aligned} P(+HD|+ECG) &= \frac{P(+HD) * P(+ECG|+HD)}{P(+ECG|+HD)*P(+HD) + P(+ECG|-HD)*P(-HD)} \\ &= \frac{(0.1) * (0.9)}{(0.9)*(0.1) + (0.05)*(0.9)} \\ &= 0.67. \end{aligned}$$

Interpretation: 2/3 of patient with chest pain produces a +ve ECG test